

The Riemann Zeta Function and the Riemann Hypothesis

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Abstract

In August 1859, Bernhard Riemann, 32 years of age, became a corresponding member of the Berlin Academy. There, it was customary to produce a publication on such occasion. He handed in a paper titled “On the Number of Primes less than a given Quantity”. This paper has been keeping Mathematicians busy, worldwide for almost 150 years now. It contains a hypothesis, known as the “Riemann Hypothesis” which is, after Fermat’s last Theorem has been proved in 1993 regarded as the greatest unsolved problem of Mathematics. The Riemann Hypothesis is: “All non-trivial zeroes of the zeta function have real part one-half!” My aim is to explain the underlying Maths, especially the Riemann Zeta Function, so that a general understanding of the Hypothesis can be achieved. Although there are “real-world” applications of the Riemann Zeta Function as well, I will focus on the theoretical properties only. A short historical overview will be given at the end as well as the original paper Riemann’s and other related material.

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1. Prime Numbers

1.1 Introduction

Definition: Consider \mathbf{N} , the set of natural numbers. A number $x \in \mathbf{N}$ that divides exactly into a number $n \in \mathbf{N}$ is called a *factor*. Every natural number has 1 and itself as factors, called *trivial factors*. All other factors, if they exist are called *proper factors*. A *prime number* is a natural number, greater than 1* that has no proper factors.

Here is a list of the first 100 prime numbers:

2	3	5	7	11	13	17	19	23	29
31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	101	103	107	109	113
127	131	137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223	227	229
233	239	241	251	257	263	269	271	277	281
283	293	307	311	313	317	331	337	347	349
353	359	367	373	379	383	389	397	401	409
419	421	431	433	439	443	449	457	461	463
467	479	487	491	499	503	509	521	523	541

(Table 1.1)

Are there infinitely many prime numbers? Euclid proved about 300 B.C. that there are.

Theorem 1.1 : There are infinitely many prime numbers.

Proof: Suppose n is a prime number. Let $m = (1 \cdot 2 \cdot 3 \cdot \dots \cdot p) + 1$. Then m does not divide exactly by any $n < p$, $n \in \mathbf{N}$. The remainder will always be 1 and $m \equiv 1 \pmod{q}$ such that $q \in \mathbf{N} \wedge q < p$. This implies that m has either no proper factors, which implies that m is a prime number itself, or its smallest proper factor is greater p . Since every natural number's smallest factor is a prime number, it follows that there are infinitely many prime numbers.

Theorem 1.2 : Any natural number can be factored as a product of one or more primes.

(can be proved by induction)

* Why is 1 not considered a prime number? 1 has no proper factors! The answer is that it is just more convenient not to consider 1 as being prime. Most of the theorems regarding natural and prime numbers would have to be rephrased to: "...for all primes greater than 1..." A similar example is the definition for factorials. Consider the

Taylor series: $\sum_0^n (x-a)^n \left(\frac{f^n(a)}{n!} \right)$ Since division by zero is not defined ($n=0$) we let $0!=1$.

1.2 The Prime Counting Function

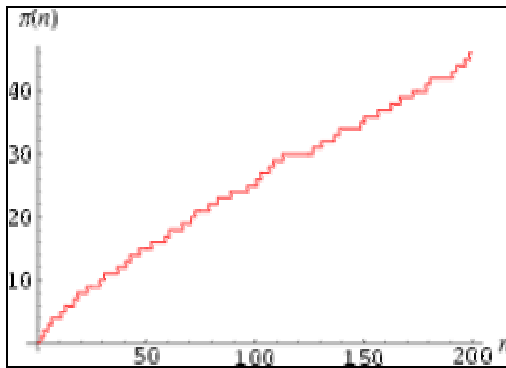
Definition 1.3 : The prime counting function denoted by $\pi(n)$ is defined to be the number of prime numbers less than or equal to n .

If we extend table 1.1 we see that between:

1 - 100 :	25 primes
401 - 500 :	17 primes
901 - 1000 :	14 primes
999901 - 1000000 :	8 primes

(Table 1.2)

The distribution of prime numbers seems to decrease. But Euclid's proof implies that it will never reach zero.



(Figure 1.3) [Wolfram Research Inc (1999)]

1.3 The Sieve of Eratosthenes

In 230 B.C. Eratosthenes of Cyrene (nowadays in Libya) discovered what is known as the Sieve of Eratosthenes, a method for finding Prime Numbers:

1. write down an ordered list of integers, starting with 2
2. remove every second number greater 2
3. let p be the next lowest number in the list
4. remove every p^{th} number greater p from the list, starting at p if it hasn't already been removed
5. repeat from 3.

After applying this procedure the list will only contain prime numbers. In fact, using this algorithm which can easily be implemented with any programming language, we are able to find all prime numbers less than $(p+1)^2$ unless they are very large, because then the computing power required would be extremely high.

Example:

2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32

(Figure 1.4)

We have found all primes less than 36! ($p=5$ at our last step)

1.4 The Prime Number Theorem

What do we know about prime numbers so far?

- We can factor each natural number as a product of prime numbers
- There are infinitely many prime numbers
- We can create a list of as many prime numbers as we want using the Sieve Method (although not feasible for very large numbers)
- We know about the prime counting function and can therefore compute the number of primes less than a given quantity (e.g. in Matlab using the command `primes(n)`)

Mathematically the prime counting function is not very satisfactory, because it is based on simply counting prime numbers. It is desirable to find an exact way to mathematically describe the *distribution of prime numbers*. The Prime Number Theorem, which exists in different versions allows us to approximate $\pi(n)$.

Theorem 1.4 : In the neighbourhood of a big number, the probability of a natural number being prime is approximately $\frac{1}{\log(n)}$.

The n^{th} prime number is approximately $n \cdot \log(n)$.

$$\pi(n) \approx \frac{n}{\log(n)}$$

An even better approximation can be achieved using the Log-Integral function.

Definition 1.5 : The Log-Integral function, denoted by $Li(x)$ is given by: $\int_0^x \frac{1}{\log(t)} dt$

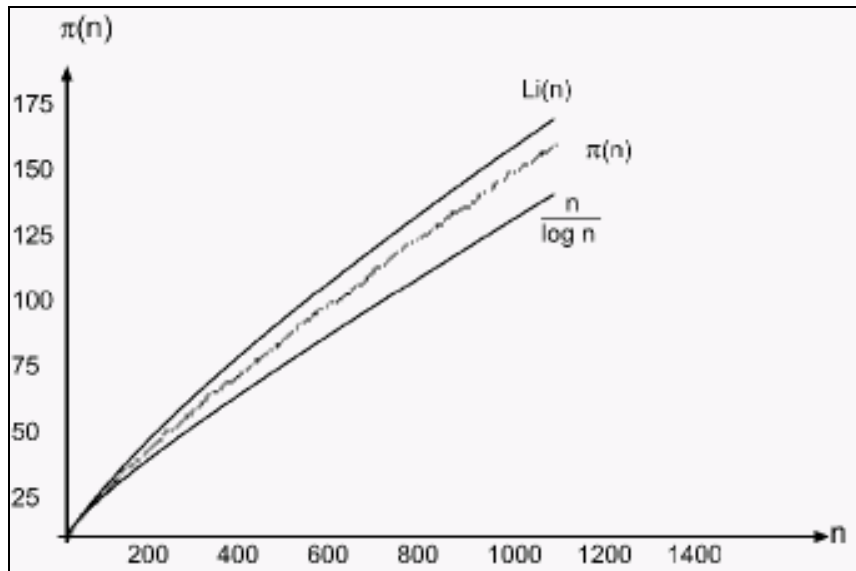
As n gets large, $Li(n) \approx \frac{n}{\log(n)}$. This implies that the prime counting function is approximately equal to the Log-Integral function for large n . That is: $\pi(n) \approx Li(n)$

This is referred to as the Prime Number Theorem as well, depending on the author.

Theorem 1.6 : The relative error in the approximation $\pi(n) \approx Li(n)$ approaches zero as x tends to infinity.*

* See (Edwards, 1974, p.76) for the proof.

The following graph shows the functions involved in the Prime Number Theorem:



(Figure 1.5)

Note that for large n $\pi(n)$ is bounded above by $Li(n)$ and bounded below by $\frac{n}{\log(n)}$.

2. Riemann's Zeta Function

2.1 Definition and the Euler Product Formula

Definition 2.1 : The Riemann Zeta function (pronounced “Zeta of s”) is the infinite series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$$

The following theorem states what is known as the *Euler Product Formula*.

Theorem 2.2 : Zeta of s is equal to the product over all primes p of $\frac{1}{(1 - p^{-s})}$.

That is:

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$

On the left hand side we have an infinite series and on the right hand side an infinite product over prime numbers. This means that prime numbers are related to the Zeta function in some way. Section 3 will reveal more details.

Proof for the *Euler Product Formula*:

$$(i) \quad \zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \dots$$

multiply both sides by $\frac{1}{2^s}$

$$(ii) \quad \frac{1}{2^s} \cdot \zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \frac{1}{12^s} + \dots$$

subtract (ii) from (i) to eliminate all even-numbered terms from the RHS

$$(iii) \quad \left(1 - \frac{1}{2^s}\right) \cdot \zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \dots$$

multiply both sides by $\frac{1}{3^s}$

$$(iv) \quad \frac{1}{3^s} \cdot \left(1 - \frac{1}{2^s}\right) \cdot \zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{27^s} + \frac{1}{33^s} + \dots$$

subtract (iv) from (iii) to eliminate all multiples of 3 from the RHS

$$(v) \quad \left(1 - \frac{1}{3^s}\right) \cdot \left(1 - \frac{1}{2^s}\right) \cdot \zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{17^s} + \frac{1}{19^s} + \dots$$

Note the similarity to the Sieve of Eratosthenes. When continuing this process, one eliminates the prime numbers with all its multiples from the RHS while they appear on the LHS of the original equation.

Now, let $p(n)$ be the n^{th} prime number. Then, as $n \rightarrow \infty$:

$$\left(1 - \frac{1}{p_n^s}\right) \cdot \left(1 - \frac{1}{p_{n-1}^s}\right) \cdot \left(1 - \frac{1}{p_{n-2}^s}\right) \cdot \dots \cdot \left(1 - \frac{1}{5^s}\right) \cdot \left(1 - \frac{1}{3^s}\right) \cdot \left(1 - \frac{1}{2^s}\right) \cdot \zeta(s) = 1$$

$$\zeta(s) = (1 - 2^{-s})^{-1} \cdot (1 - 3^{-s})^{-1} \cdot (1 - 5^{-s})^{-1} \cdot \dots \cdot (1 - p_{n-2}^{-s})^{-1} \cdot (1 - p_{n-1}^{-s})^{-1} \cdot (1 - p_n^{-s})^{-1}$$

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}$$

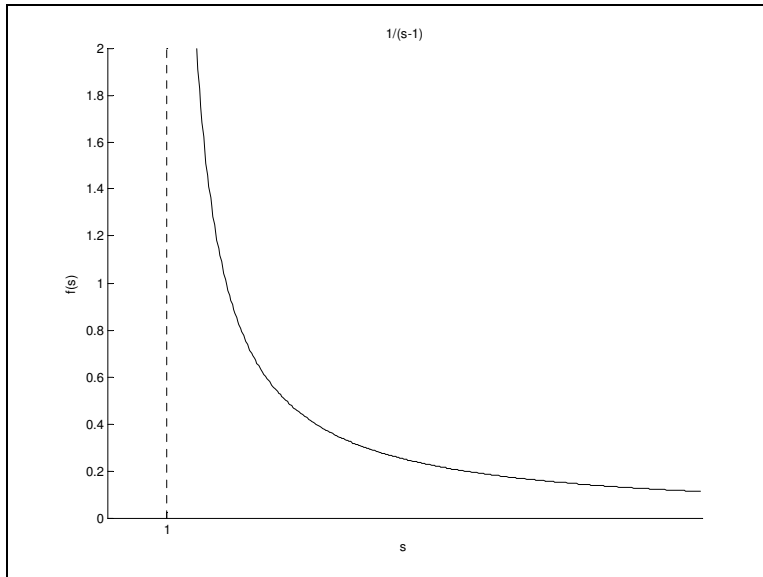
q.e.d.

2.2 Plotting the Zeta function for $s > 1$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$$

Consider the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ as special cases of the Zeta function. Since the p-series is known to converge for $p > 1$, it follows that the Zeta function converges for $s > 1$ as well.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^s} &\Leftrightarrow N \rightarrow \infty, \int_1^N x^{-s} dx = \left[\frac{x^{-s+1}}{-s+1} \right]_1^N = \left[\frac{1}{x^{-s}} \cdot \frac{1}{-s+1} \right]_1^N = \frac{1}{N^{s-1}} \cdot \left(\frac{1}{-s+1} \right) - \frac{1}{1} \cdot \left(\frac{1}{-s+1} \right) \\ &= \frac{1}{s-1} \cdot \left[1 - \frac{1}{N^{s-1}} \right] = \frac{1}{s-1} \text{ if } s > 1 \text{ as } N \rightarrow \infty \end{aligned}$$



(Figure 2.1)

Note the asymptotic behaviour at $s=1$. For $s=1$ the Zeta function reduces to the harmonic series which is divergent.

2.3 Plotting the Zeta function for $s < 1$

The p-series is defined for real numbers only and is only a special case of the Zeta Function. The Zeta function is a complex valued function. Its domain is therefore not confined to the real numbers. But before we consider the complex plane, let's look at the values of $\zeta(s)$ for $0 < s < 1$ and real negative numbers (for which the p-series is divergent).

How can we “extend” the domain to values between zero and one? A process called *analytic continuation* is required.

Consider another function first:

$$\begin{aligned}\text{Let } \eta(s) &= 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s} + \dots \\ &= (1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \dots) - 2 \cdot (\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \dots)\end{aligned}$$

$$[A - B + C - D = (A + B + C + D) - 2 \cdot (B + D)]$$

Remember from Calculus II that this kind of algebraic manipulation of alternating infinite series is only possible if the series is absolutely convergent. Hence we need to restrict ourselves to apply this manipulation to s -values greater zero only (because $\eta(s)$ is only absolutely convergent for $s > 1$).

$$\eta(s) = (1 - 2 \cdot \frac{1}{2^s}) \cdot \zeta(s)$$

$$\text{Then } \zeta(s) = \frac{\eta(s)}{1 - \frac{1}{2^{s-1}}}, \text{ which we can evaluate for: } s > 0$$

[based on Derbyshire (2003)]

The expression on the right hand side is defined for all $s > 0, s \neq 1$, because the series defining $\eta(s)$ converges for $s > 0$ by the alternating series test. We take this as the definition for $\zeta(s)$ $s > 0$.

Now we can compute the Zeta function for all positive real numbers not equal to one.

To extend the domain of the Zeta function to negative values requires more sophisticated techniques. See Appendix A for further details on the following theorem.

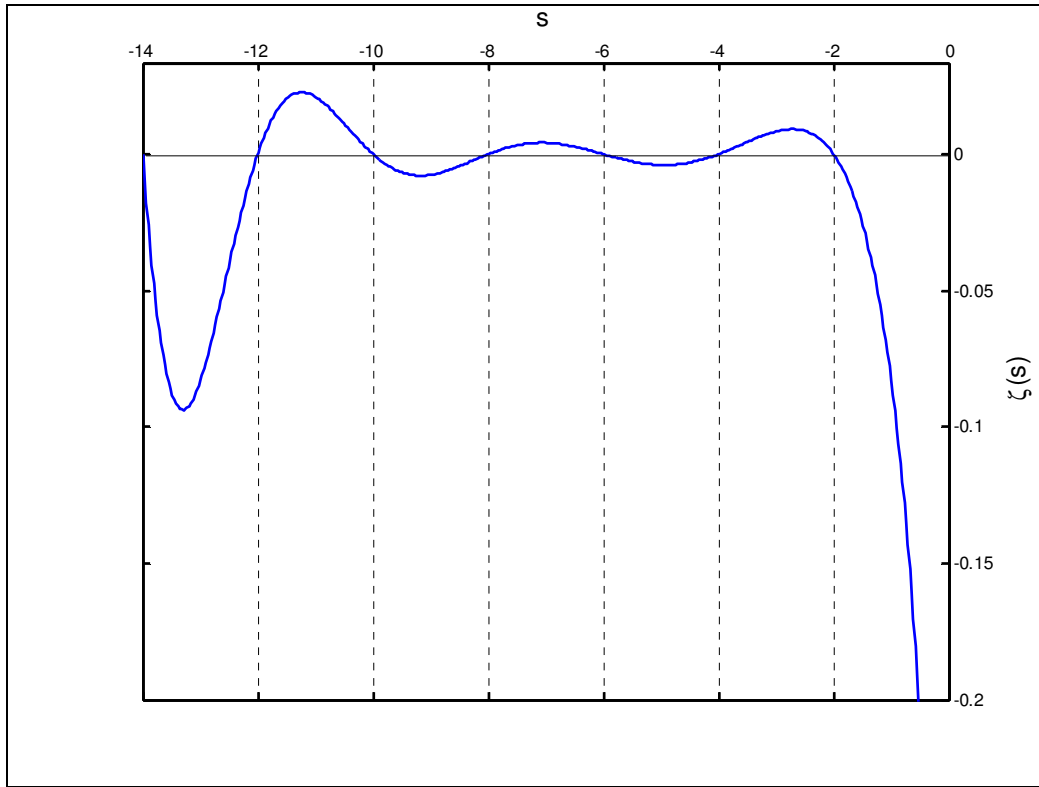
In 1749 Euler suggested a formula which Riemann proved in his 1859 Paper that relates $\zeta(s)$ and $\zeta(1-s)$, known as *the functional equation of the zeta function*.

Theorem 2.3 : The functional equation of the zeta function is given by:

$$\zeta(1-s) = 2^{1-s} \cdot \pi^{-s} \cdot \sin\left(\frac{1-s}{2} \cdot \pi\right) \cdot (s-1)! \cdot \zeta(s)$$

This formula allows us to compute values of the Zeta function for negative values of s . For example, to compute $\zeta(-15)$, plug in $s = 16$ on the RHS of the equation.

Here is a partial plot of the Zeta function for negative values of s:



(Figure 2.2)

Theorem 2.4 : All negative even integers are *zeroes* of the Zeta function.

Proof: (informal) To calculate negative even integers of the Zeta function, one has to plug in odd positive integers greater one into the functional equation. Then the term $\sin\left(\frac{1-s}{2} \cdot \pi\right)$ will become zero, since the sinus of every even multiple of π is equal to zero.

These *zeroes* of the Zeta function are called *trivial zeroes*. The Riemann Hypothesis involves *non-trivial zeroes* which will be covered in the next section.

3. The Riemann Hypothesis

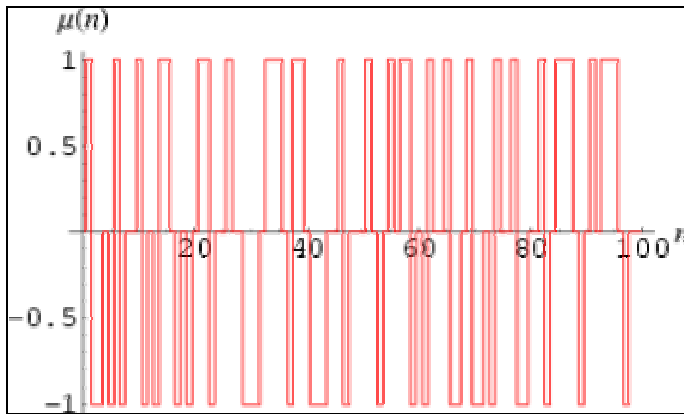
“All non-trivial zeroes of the zeta function have real part one-half!”

This section will combine some of the results of sections one and two. In order to gain a basic understanding of the hypothesis some additional definitions and methods are introduced in the next sub-section. Proofs and detailed explanations of most of the following are beyond the scope of this report and are omitted.

3.1 The underlying Mathematics

Definition 3.1 : The Moebius Function is defined by

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ has one or more repeated prime factors} \\ 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes} \end{cases}$$



(Figure 3.1)

[Wolfram Research Inc (1999)]

The Zeta function is related to the Moebius function by:

$$\frac{1}{\zeta(s)} = \sum_n \frac{\mu(n)}{n^s}$$

Let $\pi(n)$ be the prime counting function defined in 1.3.

Definition 3.2 : Let $J(x)$ be defined by

$$J(x) = \pi(x) + \frac{1}{2}\pi(\sqrt[2]{x}) + \frac{1}{3}\pi(\sqrt[3]{x}) + \frac{1}{4}\pi(\sqrt[4]{x}) + \dots$$

Note that the RHS of this equation does not represent an infinite sum, because as soon as $\sqrt[n]{x} < 2 \rightarrow \pi(\sqrt[n]{x}) = 0$.

Using a process called *Moebius Inversion* it is possible to express $\pi(x)$ in terms of $J(x)$. Then,

$$\pi(x) = J(x) - \frac{1}{2}J(\sqrt[2]{x}) - \frac{1}{3}J(\sqrt[3]{x}) - \frac{1}{5}J(\sqrt[5]{x}) + \frac{1}{6}J(\sqrt[6]{x}) - \frac{1}{7}J(\sqrt[7]{x}) + \frac{1}{10}J(\sqrt[10]{x}) - \dots$$

Applying the Moebius Function gives:

$$\pi(x) = \sum_n \frac{\mu(n)}{n} \cdot J(\sqrt[n]{x})$$

It is also possible to express $J(x)$ in terms of the Zeta function:

$$\frac{1}{s} \log \zeta(s) = \int_0^\infty J(x) \cdot x^{-s-1} dx$$

We can express the prime counting function in terms of $J(x)$ and we can express $J(x)$ in terms of the Zeta function. Considering that, we can express the prime counting function in terms of the Zeta function. This means that a close study of the Zeta function will reveal the nature of the distribution of prime numbers. This is one of the remarkable results of Riemann's paper. It combines the domain of *Number Theory* (the prime counting function) with the domain of *Analysis* (the Zeta function) to a branch of Mathematic called *Analytic Number Theory*.

3.2 Non-trivial Zeroes

As we have seen in the last sub-section, the Zeta function is related to the prime counting function, because both are related to $J(x)$. Let's therefore have a closer look at this function. It can also be expressed in the following way:

[see Appendix C for more details]

$$J(x) = Li(x) - \sum_{\beta} Li(x^{\beta}) - \log 2 + \int_x^{\infty} \frac{dt}{t \cdot (t^2 - 1) \cdot \log t}$$

Let's analyse the single terms of the above expression:

- $Li(x)$ is the log-integral function defined in 1.4
- $\log 2$ is just a number

$$- \int_x^{\infty} \frac{dt}{t \cdot (t^2 - 1) \cdot \log t}$$

gets very small for large x , referring to prime distributions for large numbers, the region of interest

The crucial term therefore is: $\sum_{\beta} Li(x^{\beta})$.

$$\text{Let } \xi(x) = \sum_{\beta} Li(x^{\beta})$$

Riemann transformed the Zeta function $\zeta(s)$ to a function involving $\xi(x)$. The trivial zeroes of the Zeta function "disappear" after this transformation. But the non-trivial zeroes of the Zeta function, which are the subject of the Riemann Hypothesis, correspond exactly to all possible roots β of $\xi(x)$. The sum in the above expression is taken over all these roots, that is over all non-trivial zeroes of the Zeta function.

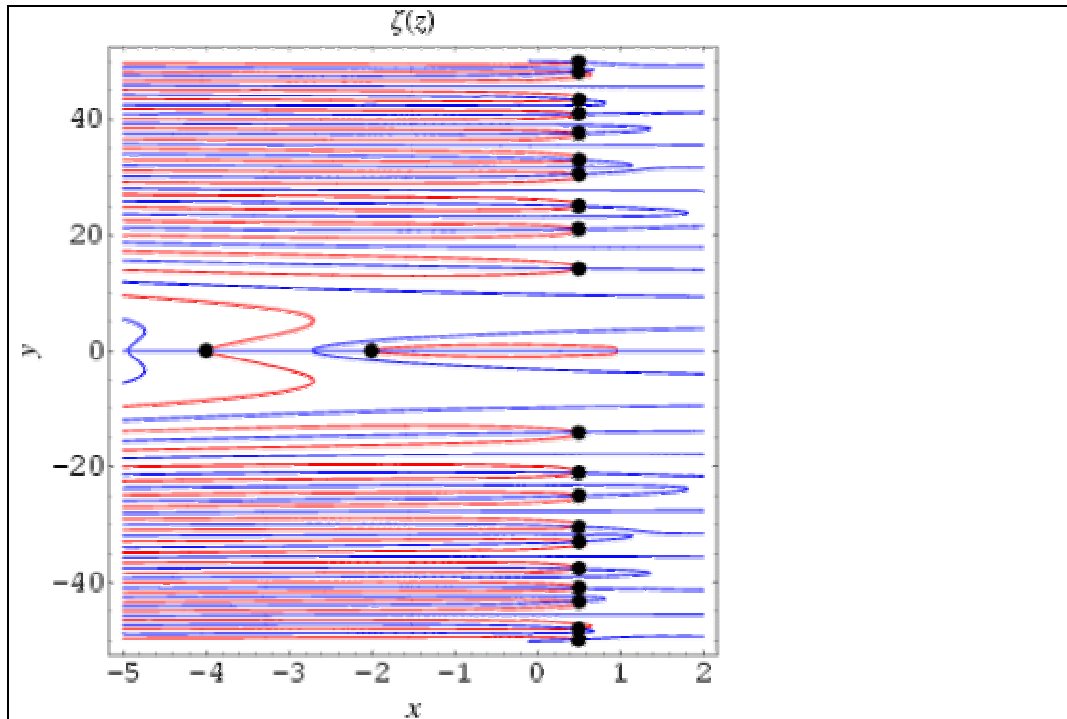
So, by the Riemann Hypothesis, if setting $\beta = \frac{1}{2} + i \cdot b$ (real part one-half), one can find all non-trivial zeroes of $\zeta(s)$, provided it is true.

[see Appendix C for more details]

In 1914 G.H. Hardy proved that the Zeta function has infinitely many non-trivial zeroes whose real part is indeed one-half. But this doesn't prove the hypothesis.

Here, having provided the main concepts for a basic understanding of the Riemann Hypothesis, I will stop the discussion and finish with the following two graphs.

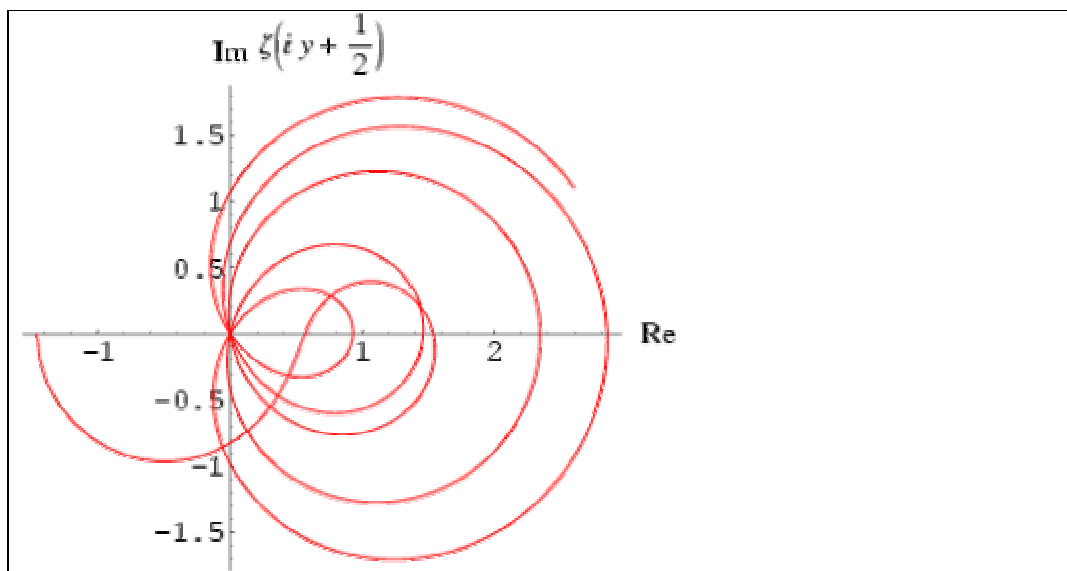
The position of the complex zeroes can be seen by plotting the contours of zero real and imaginary parts, as illustrated below. The zeroes (indicated dots) occur where the curves intersect.



(Figure 3.2)

[Wolfram Research Inc (1999)]

A plot of the Zeta functions for s on the *critical line*: $s = \frac{1}{2} + i \cdot b$



(Figure 3.3)

[Wolfram Research Inc (1999)]

The Functional Equation of the Zeta Function

To derive the Functional Equation of the Zeta Function (given in section 2.3) which can be used to evaluate the Zeta Function for negative s-values, it is necessary to introduce some further techniques and special functions first. It involves the Γ -Function and a special case of the θ -Functions which in turn uses Poisson Summation. I wish to acknowledge the help of Dr. Shaun Cooper for this part.

Poisson Summation Formula:

Suppose $f(x)$ is a function which decays sufficiently rapidly as $x \rightarrow \pm\infty$. Then f is smaller than $\frac{1}{|x|^{1+\delta}}$ for some $\delta > 0$ for large $|x|$.

$$\text{Let } F(x) = \sum_{n=-\infty}^{\infty} f(x+n).$$

$$\Rightarrow F(x+1) = \sum_{n=-\infty}^{\infty} f(x+1+n) \quad \text{Let } (n+1) = k$$

$$\Rightarrow F(x+1) = \sum_{k=-\infty}^{\infty} f(x+k) = F(x)$$

$\Rightarrow F$ is periodic with period one and therefore has a Fourier Series expansion:

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{-2\pi i k x}$$

where $c_k = \int_{-\infty}^{\infty} f(t) \cdot e^{-2\pi i k t} dt$

θ -Functions:

Let $f(x) = e^{-sx^2\pi}$, $s > 0$. Then $f(x)$ decays sufficiently rapidly. After applying

Poisson Summation: $\sum_{n=-\infty}^{\infty} e^{-s(x+n)^2\pi} = \sum_{k=-\infty}^{\infty} c_k \cdot e^{2\pi i k x}$, where $c_k = \frac{1}{\sqrt{s}} \cdot e^{-\frac{\pi k^2}{s}}$

Now consider the θ -Function with $x = 0$.

$$\text{Then } \sum_{n=-\infty}^{\infty} e^{-s(x+n)^2\pi} = \sum_{n=-\infty}^{\infty} e^{-s \cdot n^2\pi} = c_k = \frac{1}{\sqrt{s}} \sum_{k=-\infty}^{\infty} e^{-\frac{\pi k^2}{s}}$$

Let

$$\varphi(s) = \sum_{n=-\infty}^{\infty} e^{-s \cdot n^2\pi}, \text{ then } \varphi\left(\frac{1}{s}\right) = \sum_{n=-\infty}^{\infty} e^{-\pi \frac{n^2}{s}}$$

$$\Rightarrow \varphi(s) = \frac{1}{\sqrt{s}} \cdot \varphi\left(\frac{1}{s}\right)$$

$$\sum_{n=-\infty}^{\infty} e^{-s \cdot n^2\pi} = 1 + 2 \cdot \sum_{n=1}^{\infty} e^{-s \cdot n^2\pi}$$

Riemann defined the following functions based on the above. These results will be used later to derive the Functional Equation of the Zeta function.

$$\text{Let } \psi(s) = \sum_{n=1}^{\infty} e^{-s \cdot n^2\pi}. \quad \text{Then,}$$

$$\varphi(s) = 1 + 2 \cdot \psi(s)$$

$$\psi(s) = \frac{\varphi(s) - 1}{2}$$

$$\psi\left(\frac{1}{s}\right) = \frac{\sqrt{s}}{2} + \sqrt{s} \cdot \psi(s) - \frac{1}{2}$$

The Γ -Function:

Definition: $\Gamma(x) = \int_0^{\infty} t^{x-1} \cdot e^{-t} dt$

Properties:

$$\Gamma(x+1) = x \cdot \Gamma(x)$$

$$\Gamma(n+1) = n!$$

$$\Gamma(1) = 1$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Reflection Formula:

$$\Gamma(x) \cdot \Gamma(1-x) = \frac{\pi}{\sin(\pi \cdot x)}$$

Duplication Formula:

$$\Gamma(2 \cdot x) \cdot \Gamma\left(\frac{1}{2}\right) = 2^{2x-1} \cdot \Gamma(x) \cdot \Gamma\left(x + \frac{1}{2}\right)$$

The Zeta Function:

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} t^{\frac{s}{2}-1} \cdot e^{-t} dt$$

Substitution: Let $t = \pi k^2 y$, $dt = \pi k^2 dy$
--

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} (\pi k^2 y)^{\frac{s}{2}-1} \cdot e^{-\pi k^2 y} \cdot \pi k^2 dy$$

$$\Gamma\left(\frac{s}{2}\right) = (\pi k^2)^{\frac{s}{2}} \cdot \int_0^{\infty} y^{\frac{s}{2}-1} \cdot e^{-\pi k^2 y} dy$$

$$\Rightarrow \frac{1}{k^2} = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \cdot \int_0^{\infty} y^{\frac{s}{2}-1} \cdot e^{-\pi k^2 y} dy$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \cdot \int_0^{\infty} y^{\frac{s}{2}-1} \cdot \sum_{k=1}^{\infty} e^{-\pi k^2 y} dy$$

$$\Rightarrow \zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \cdot \int_0^{\infty} y^{\frac{s}{2}-1} \cdot \psi(y) \cdot dy$$

$$\Rightarrow \pi^{\frac{-s}{2}} \cdot \Gamma\left(\frac{s}{2}\right) \cdot \zeta(s) = \int_0^1 y^{\frac{s}{2}} \cdot \psi(y) \cdot \frac{dy}{y} + \int_1^{\infty} y^{\frac{s}{2}} \cdot \psi(y) \cdot \frac{dy}{y}$$

Substitution: Let

$$y = \frac{1}{x} \Rightarrow x \cdot y = 1$$

$$dy = -\frac{1}{x^2} dx \Rightarrow \frac{dy}{y} = -\frac{1}{x^2 y} dx = -\frac{1}{x} dx$$

$$\text{Limits : } y = 1 \Rightarrow x = 1$$

$$y = 0 \Rightarrow x \rightarrow \infty$$

$$\begin{aligned} \text{Then, } \pi^{-\frac{s}{2}} \cdot \Gamma\left(\frac{s}{2}\right) \cdot \zeta(s) &= \int_0^\infty x^{-\frac{s}{2}} \cdot \psi\left(\frac{1}{x}\right) \cdot \frac{dx}{x} + \int_1^\infty x^{\frac{s}{2}} \cdot \psi(x) \cdot \frac{dx}{x} \\ &= \int_1^\infty x^{-\frac{s}{2}} \cdot \left[\sqrt{x} \cdot \psi(x) + \frac{\sqrt{x}}{2} - \frac{1}{2} \right] \frac{dx}{x} + \int_1^\infty x^{\frac{s}{2}} \cdot \psi(x) \cdot \frac{dx}{x} \\ &= \frac{1}{2} \cdot \int_1^\infty x^{\frac{1-s}{2}-1} dx - \frac{1}{2} \cdot \int_1^\infty x^{\frac{1}{2}-1} dx + \int_1^\infty \psi(x) \cdot \left[x^{\frac{s}{2}} + x^{\frac{(1-s)}{2}} \right] \frac{dx}{x} \\ &= \frac{1}{2} \cdot \frac{-1}{\left(\frac{1}{2} - \frac{s}{2}\right)} - \frac{1}{2} \cdot \frac{-1}{-\frac{s}{2}} + \int_1^\infty \psi(x) \cdot \left[x^{\frac{s}{2}} + x^{\frac{(1-s)}{2}} \right] \frac{dx}{x} \\ &= \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \psi(x) \cdot \left[x^{\frac{s}{2}} + x^{\frac{(1-s)}{2}} \right] \frac{dx}{x} \end{aligned}$$

$$\Rightarrow \pi^{-\frac{s}{2}} \cdot \Gamma\left(\frac{s}{2}\right) \cdot \zeta(s) + \frac{1}{1-s} + \frac{1}{s} = \int_1^\infty \psi(x) \cdot \left[x^{\frac{s}{2}} + x^{\frac{(1-s)}{2}} \right] \frac{dx}{x}$$

Now, let

$$F(s) = \pi^{\frac{s}{2}} \cdot \Gamma\left(\frac{s}{2}\right) \cdot \zeta(s)$$

$$G(s) = \frac{1}{1-s} + \frac{1}{s}$$

$$H(s) = \int_1^{\infty} \psi(x) \cdot \left[x^{\frac{s}{2}} + x^{\frac{(1-s)}{2}} \right] \frac{dx}{x}$$

Note that,

$$G(1-s) = G(s)$$

$$H(1-s) = H(s)$$

$$F(s) = H(s) - G(s)$$

$$\Rightarrow F(1-s) = H(1-s) - G(1-s) = H(s) - G(s) = F(s)$$

$$\Rightarrow \pi^{\frac{-(1-s)}{2}} \cdot \Gamma\left(\frac{1-s}{2}\right) \cdot \zeta(1-s) = \pi^{\frac{s}{2}} \cdot \Gamma\left(\frac{s}{2}\right) \cdot \zeta(s)$$

This is one version of the functional equation of the Zeta function relating $\zeta(s)$ and $\zeta(1-s)$. To change it to the form given in section 2.3 the reflection formula and the duplication formula for the gamma function need to be applied.

$$\pi^{\frac{-(1-s)}{2}} \cdot \Gamma\left(\frac{1-s}{2}\right) \cdot \zeta(1-s) = \pi^{\frac{s}{2}} \cdot \Gamma\left(\frac{s}{2}\right) \cdot \zeta(s)$$

$$\begin{aligned} \Rightarrow \zeta(1-s) &= \pi^{-s} \cdot \zeta(s) \cdot \pi^{\frac{1}{2}} \cdot \Gamma\left(\frac{s}{2}\right) \cdot \left[\Gamma\left(\frac{1-s}{2}\right)\right]^{-1} \\ &= \pi^{-s} \cdot \sin\left(\frac{1-s}{2} \cdot \pi\right) \cdot \zeta(s) \cdot \pi^{\frac{1}{2}} \cdot \Gamma\left(\frac{s}{2}\right) \cdot \Gamma\left(\frac{1+s}{2}\right) && \text{by the reflection formula} \\ &= \pi^{-s} \cdot \sin\left(\frac{1-s}{2} \cdot \pi\right) \cdot \zeta(s) \cdot \pi^{\frac{1}{2}} \cdot 2^{1-s} \cdot \Gamma\left(\frac{1}{2}\right) \cdot \Gamma(s) && \text{by the duplication formula} \\ &= 2^{1-s} \cdot \pi^{-s} \cdot \sin\left(\frac{1-s}{2} \cdot \pi\right) \cdot (s-1)! \zeta(s) \end{aligned}$$

Bernhard Riemann and Mathematics in the 19th Century

Bernhard Riemann is a Mathematician from the nineteenth Century. He lived in a period considered the most important period of modern Mathematics in which the most fruitful and remarkable discoveries and developments had taken place.

Moritz Cantor's *Die Geschichte der Mathematik* is probably the most extensive history of Mathematics book, consisting of three volumes summing up to approximately 3600 pages. This work ends with the year 1799 but consists of everything dating back to the ancient Egypt, about 2000 B.C. where Mathematics first evolved. To tell the story of Mathematics of the time after 1800 is believed to take about 20 volumes of similar size, that is about 17000 pages [Bell, 1937], even though people like Newton, Euler and Leibniz died before that time.

Riemann was born in 1826 in Hannover, Germany. As a pupil he focussed on studying classical subjects like Theology, Hebrew and Philosophy. But soon his talent for Mathematics had been discovered by the director of the Gymnasium Riemann attended. From him he borrowed Legendre's *Theorie des Nombres* (859 pages) and returned it after 6 days saying "*That is certainly a wonderful book. I have mastered it*" [Bell, 1937].

In 1846 Riemann enrolled at the University of Goettingen to study Theology following the encouragements of his father. However, he attended lectures in Mathematics from the first term on and finally changed to the Faculty of Mathematics with his father's permission. There, among his lecturers were Gauss and Stern. [Turnbull 1998]

In 1847 Riemann moved to Berlin where he studied under Jacobi, Eisenstein and Dirichlet. Influenced particularly by Eisenstein, he studied the theory of complex variables and Elliptic functions. In 1849 Riemann returned to Goettingen to study towards his Doctorate, supervised by Gauss. He submitted his thesis about complex variables in 1851 in which he developed the idea of what is nowadays known as *Riemann Surfaces*. Gauss' described Riemann after the examination of his thesis as having a "... gloriously fertile originality" [Turnbull 1998]

In the following three years Riemann worked for his *habilitation*, a thesis leading to a professorship at German Universities required for lecturing. He submitted his thesis in 1854 and held a lecture as part of this process "*On the hypotheses that lie at the foundations of geometry*" Gauss was apparently the only one in the audience who could appreciate the depth of Riemann's work. [Turnbull 1998]

In 1859 Riemann was appointed to the chair of Mathematics in Goettingen and shortly after that he was elected to the Berlin Academy where he handed in his famous paper *On the Number of Primes less than a given Quantity*. Among his supporters in Berlin was K.T.W. Weierstrass.

Riemann died in 1866 only 40 years of age.

A poem on the zeta function

Where are the zeros of zeta of s ?
G.F.B. Riemann has made a good guess;
They're all on the critical line, saith he,
And their density's one over $2 \pi \log t$.

This statement of Riemann's has been like a trigger,
And many good men, with vim and with vigour,
Have attempted to find, with mathematical rigour,
What happens to zeta as $\text{mod } t$ gets bigger.

The efforts of Landau and Bohr and Cramer,
Littlewood, Hardy and Titchmarsh are there,
In spite of their effort and skill and finesse,
In locating the zeros there's been little success.

In 1914 G.H. Hardy did find,
An infinite number do lay on the line,
His theorem, however, won't rule out the case,
There might be a zero at some other place.

Oh, where are the zeros of zeta of s ?
We must know exactly, we cannot just guess.
In order to strengthen the prime number theorem,
The integral's contour must never go near 'em.

Let P be the function p minus Li ,
The order of P is not known for x high,
If square root of x times $\log x$ we could show,
Then Riemann's conjecture would surely be so.

Related to this is another enigma,
Concerning the Lindelöf function μ sigma.
Which measures the growth in the critical strip,
On the number of zeros it gives us a grip.

But nobody knows how this function behaves,
Convexity tells us it can have no waves,
Lindelöf said that the shape of its graph,
Is constant when sigma is more than one-half.

There's a moral to draw from this sad tale of woe,
which every young genius among you should know:
If you tackle a problem and seem to get stuck,
Use R.M.T., and you'll have better luck.

Words by Tom Apostol

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